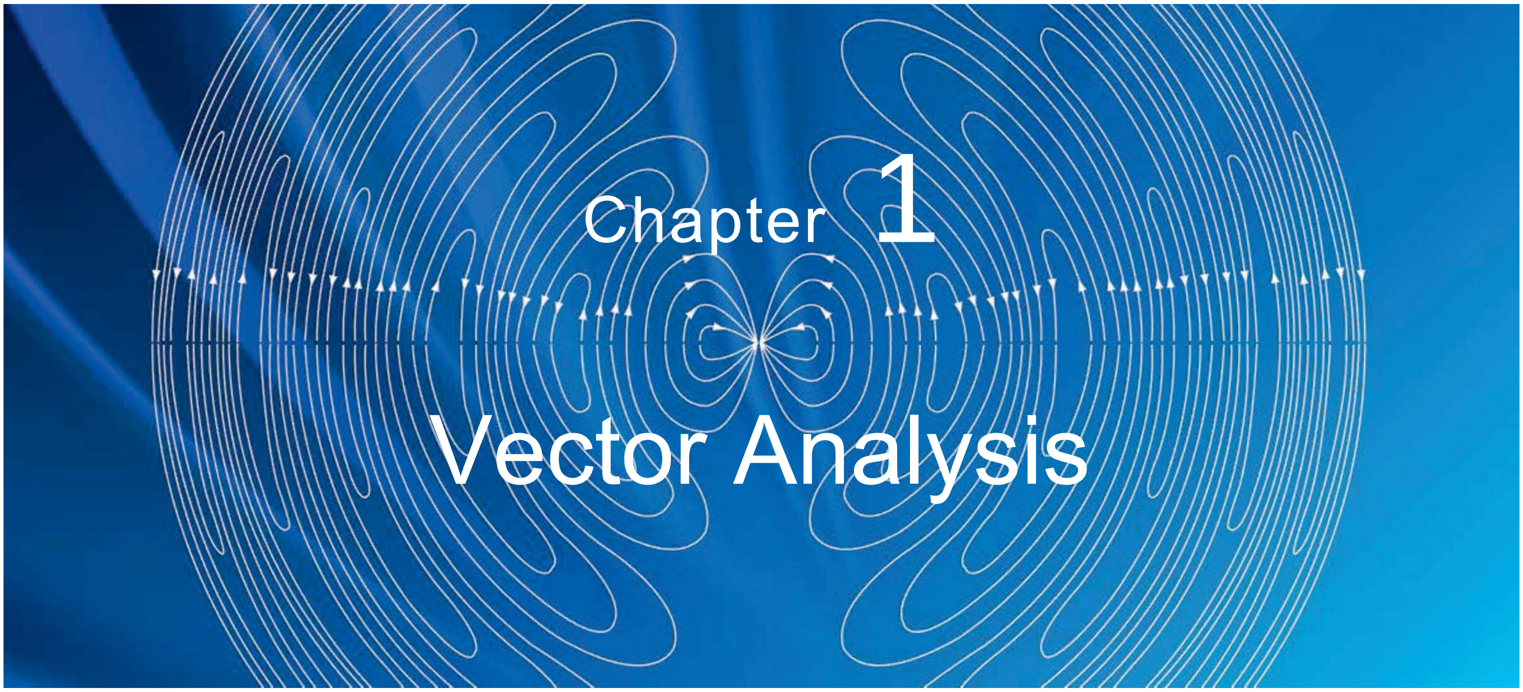


Chapter 1

Vector Analysis

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.
2. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.
3. Apply the divergence theorem and Stokes's theorem.

Overview

In our examination of wave propagation on a transmission line in Chapter 2, the primary quantities we worked with were voltage, current, impedance, and power. Each of these is a ***scalar*** quantity, meaning that it can be completely specified by its magnitude if it is a positive real number or by its magnitude and phase angle if it is a negative or a complex number (a negative number has a positive magnitude and a phase angle of π (rad)). This chapter is concerned with vectors. A ***vector*** has a magnitude and a direction. The speed of an object is a scalar, whereas its velocity is a vector.

Starting in the next chapter and throughout the succeeding chapters in this book, the primary electromagnetic quantities we deal with are the electric and magnetic fields, **E** and **H**. These, and many other related quantities, are vectors. Vector analysis provides the mathematical tools necessary for expressing and manipulating vector quantities in an efficient and convenient manner. To specify a vector in three-dimensional space, it is necessary to specify its components along each of the three directions.

► Several types of coordinate systems are used in the study of vector quantities, the most common being the Cartesian (or rectangular), cylindrical, and spherical systems. A particular coordinate system is usually chosen to best suit the geometry of the problem under consideration. ◄

Vector algebra governs the laws of addition, subtraction, and "multiplication" of vectors. The rules of vector algebra and vector representation in each of the aforementioned orthogonal coordinate systems (including vector transformation between them) are two of the three major topics treated in this chapter. The third topic is ***vector calculus***, which encompasses the laws of differentiation and integration of vectors, the use of special vector operators (gradient, divergence, and curl), and the application of certain theorems that are particularly useful in the study of electromagnetics, most notably the divergence and Stokes's theorems.

3-1 Basic Laws of Vector Algebra

A vector is a mathematical object that resembles an arrow. Vector **A** in **Fig. 3-1** has ***magnitude*** (or length) $A = |\mathbf{A}|$ and ***unit vector*** $\hat{\mathbf{a}}$:

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A. \quad (3.1)$$

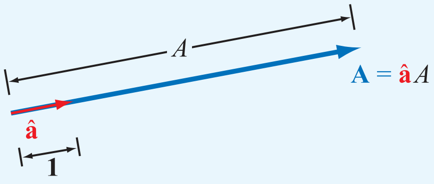
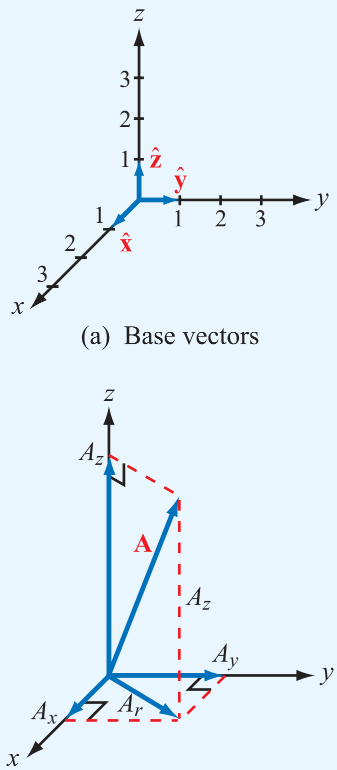


Figure 3-1 Vector $\mathbf{A} = \hat{\mathbf{a}}A$ has magnitude $A = |\mathbf{A}|$ and points in the direction of unit vector $\hat{\mathbf{a}} = \mathbf{A}/A$.



(b) Components of A

Figure 3-2 Cartesian coordinate system: (a) base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ and (b) components of vector **A**.

The unit vector $\hat{\mathbf{a}}$ has a magnitude of one ($|\hat{\mathbf{a}}| = 1$) and points from **A**'s tail or anchor to its head or tip. From Eq. (3.1),

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}. \quad (3.2)$$

In the Cartesian (or rectangular) coordinate system shown in **Fig. 3-2(a)**, the x, y, and z coordinate axes extend along directions of the three mutually perpendicular unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, which are also called ***base vectors***. The vector $\mathbf{A}$ in Fig. 3-2(b) may be decomposed as

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.3}$$

where A_x, A_y, and A_z are $\mathbf{A}$'s scalar components along the x-, y-, and z axes, respectively. The component A_z is equal to the perpendicular projection of $\mathbf{A}$ onto the z axis, and similar definitions apply to A_x and A_y. Application of the Pythagorean theorem—first to the right triangle in the x–y plane to express the hypotenuse A_r in terms of A_x and A_y and then again to the vertical right triangle with sides A_r and A_z and hypotenuse A—yields the following expression for the magnitude of $\mathbf{A}$:

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}. \tag{3.4}$$

Since A is a nonnegative scalar, only the positive root applies. From Eq. (3.2), the unit vector $\hat{\mathbf{a}}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}. \tag{3.5}$$

Occasionally, we use the shorthand notation $\mathbf{A} = (A_x, A_y, A_z)$ to denote a vector with components A_x, A_y, and A_z in a Cartesian coordinate system.

3-1.1 Equality of Two Vectors

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if they have equal magnitudes and identical unit vectors. Thus, if

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.6a}$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \tag{3.6b}$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

▶ Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another. ◀

3-1.2 Vector Addition and Subtraction

The sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ is a vector

$$\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z,$$

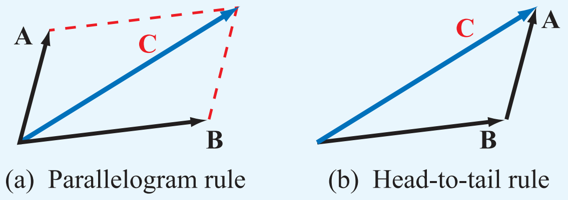


Figure 3-3 Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

given by

$$\begin{aligned}\mathbf{C} = \mathbf{A} + \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) + (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)\\ &= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z)\\ &= \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z,\end{aligned} \tag{3.7}$$

with $C_x = A_x + B_x$, etc.

▶ Vector addition is commutative:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{3.8}$$

Graphically, vector addition can be accomplished by either the parallelogram or the head-to-tail rule (Fig. 3-3). Vector $\mathbf{C}$ is the diagonal of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$. With the head-to-tail rule, we may either add $\mathbf{A}$ to $\mathbf{B}$ or $\mathbf{B}$ to $\mathbf{A}$. When $\mathbf{A}$ is added to $\mathbf{B}$, it is repositioned so that its tail starts at the tip of $\mathbf{B}$ while keeping its length and direction unchanged. The sum vector $\mathbf{C}$ starts at the tail of $\mathbf{B}$ and ends at the tip of $\mathbf{A}$.

Subtraction of vector $\mathbf{B}$ from vector $\mathbf{A}$ is equivalent to the addition of $\mathbf{A}$ to negative $\mathbf{B}$. Thus,

$$\begin{aligned}\mathbf{D} &= \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})\\ &= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z).\end{aligned} \tag{3.9}$$

Graphically, the same rules used for vector addition are also applicable to vector subtraction; the only difference is that the arrowhead of $(-\mathbf{B})$ is drawn on the opposite end of the line segment representing the vector $\mathbf{B}$ (i.e., the tail and head are interchanged).

3-1.3 Position and Distance Vectors

The ***position vector*** of a point P in space is the vector from the origin to P. Assuming points P_1 and P_2 are at (x_1, y_1, z_1) and (x_2, y_2, z_2) in Fig. 3-4, their position vectors are

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1, \tag{3.10a}$$

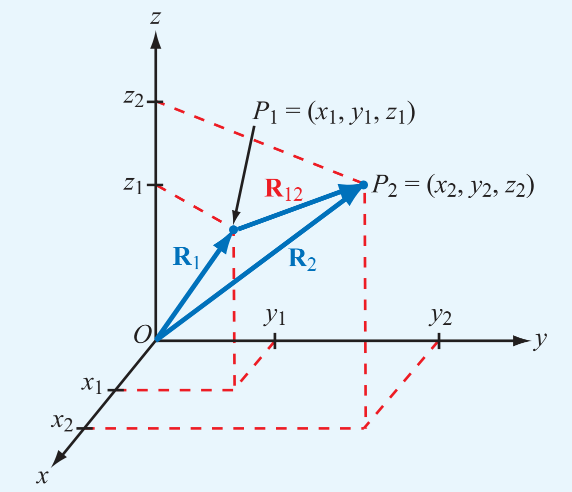


Figure 3-4 Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2, \qquad (3.10b)$$

where point O is the origin.

The ***distance vector*** from P_1 to P_2 is defined as

$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1\\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1), \qquad (3.11)\end{aligned}$$

and the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \qquad (3.12)$$

Note that the first and second subscripts of $\mathbf{R}_{12}$ denote the locations of its tail and head, respectively (**Fig. 3-4**).

3-1.4 Vector Multiplication

There exist three types of products in vector calculus: the simple product, the scalar (or dot) product, and the vector (or cross) product.

Simple Product

The multiplication of a vector by a scalar is called a ***simple product***. The product of the vector $\mathbf{A} = \hat{\mathbf{a}}A$ by a scalar k results in a vector $\mathbf{B}$ with magnitude $B = kA$ and direction the same as $\mathbf{A}$. That is, $\hat{\mathbf{b}} = \hat{\mathbf{a}}$. In Cartesian coordinates,

$$\begin{aligned}\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}kA &= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)\\ &= \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z. \qquad (3.13)\end{aligned}$$

Scalar or Dot Product

The ***scalar*** (or ***dot***) ***product*** of two co-anchored vectors $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A}\cdot\mathbf{B}$ and pronounced "A dot B," is defined geometrically as the product of the magnitude of $\mathbf{A}$ and the scalar component of $\mathbf{B}$ along $\mathbf{A}$, or vice versa. Thus,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}, \qquad (3.14)$$

where θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$ (**Fig. 3-5**) measured *from* the tail of $\mathbf{A}$ *to* the tail of $\mathbf{B}$. Angle θ_{AB} is assumed to be in the range $0 \le \theta_{AB} \le 180°$. The scalar product of $\mathbf{A}$ and $\mathbf{B}$ yields a scalar whose magnitude is less than or equal to the products of their magnitudes (equality holds when $\theta_{AB} = 0$) and whose sign is positive if $0 < \theta_{AB} < 90°$ and negative if $90° < \theta_{AB} < 180°$. When $\theta_{AB} = 90°$, $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, and their dot product is zero. The quantity $A\cos\theta_{AB}$ is the scalar component of $\mathbf{A}$ along $\mathbf{B}$. Similarly, $B\cos\theta_{BA}$ is the scalar component of $\mathbf{B}$ along $\mathbf{A}$.

The dot product obeys both the commutative and distributive properties of multiplication:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}, \qquad (3.15a)$$

(commutative property)

$$\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}. \qquad (3.15b)$$

(distributive property)

The commutative property follows from Eq. (3.14) and the fact that $\theta_{AB} = \theta_{BA}$. The distributive property expresses the fact that the scalar component of the sum of two vectors along a third one equals the sum of their respective scalar components.

The dot product of a vector with itself gives

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A^2, \qquad (3.16)$$

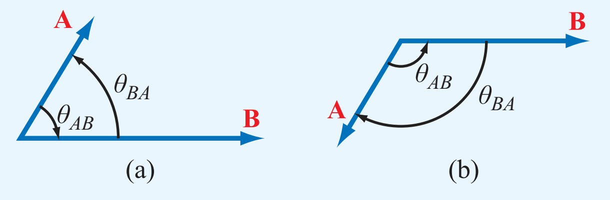


Figure 3-5 The angle θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$, measured from $\mathbf{A}$ to $\mathbf{B}$ between vector tails. The dot product is positive if $0 \le \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \le 180°$, as in (b).

which implies that

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}. \qquad (3.17)$$

Also, θ_{AB} can be determined from

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\ \sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]. \qquad (3.18)$$

Since the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are each orthogonal to the other two, it follows that

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \qquad (3.19a)$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0. \qquad (3.19b)$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z). \qquad (3.20)$$

Use of Eqs. (3.19a) and (3.19b) in Eq. (3.20) leads to

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \qquad (3.21)$$

Vector or Cross Product

The ***vector*** (or ***cross***) ***product*** of two vectors **A** and **B**, denoted $\mathbf{A} \times \mathbf{B}$ and pronounced "*A* cross *B*," yields a vector defined as

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}, \qquad (3.22)$$

where $\hat{\mathbf{n}}$ is a *unit vector normal to the plane containing* **A** *and* **B** (**Fig. 3-6(a)**). The magnitude of the cross product, $AB|\sin\theta_{AB}|$, equals the area of the parallelogram defined by the two vectors. The direction of $\hat{\mathbf{n}}$ is governed by the ***right-hand rule*** (**Fig. 3-6(b)**): $\hat{\mathbf{n}}$ points in the direction of the right thumb when the fingers rotate from **A** to **B** through the angle θ_{AB}. Note that, since $\hat{\mathbf{n}}$ is perpendicular to the plane containing **A** and **B**, $\mathbf{A} \times \mathbf{B}$ is perpendicular to both vectors **A** and **B**.

The cross product is anticommutative and distributive:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \textbf{(anticommutative)}. \qquad (3.23a)$$

The anticommutative property follows from the application of the right-hand rule to determine $\hat{\mathbf{n}}$. The distributive property follows from the fact that the area of the parallelogram formed by **A** and $(\mathbf{B} + \mathbf{C})$ equals the sum of those formed by (**A** and **B**) and (**A** and **C**):

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \quad \textbf{(distributive)} \qquad (3.23b)$$

The cross product of a vector with itself vanishes. That is,

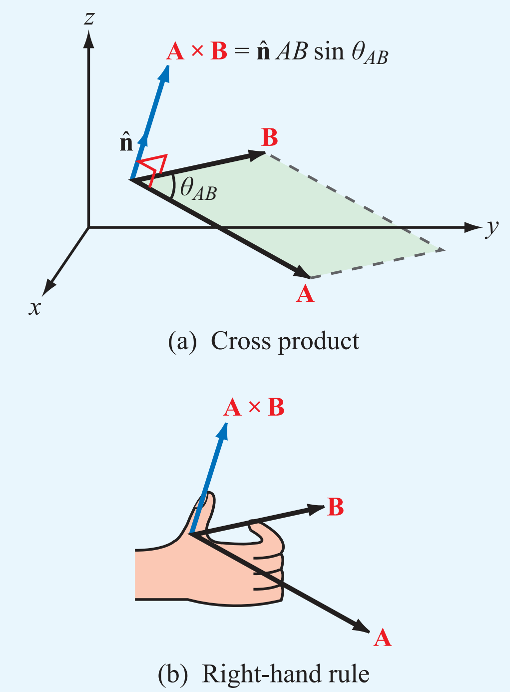
$$\mathbf{A} \times \mathbf{A} = 0. \qquad (3.24)$$

From the definition of the cross product given by Eq. (3.22), it is easy to verify that the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ of the Cartesian coordinate system obey the right-hand cyclic relations:

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \qquad (3.25)$$

Note the cyclic order $(xyzxyz\ldots)$. Also,

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \qquad (3.26)$$



(a) Cross product

(b) Right-hand rule

Figure 3-6 Cross product $\mathbf{A} \times \mathbf{B}$ points in the direction $\hat{\mathbf{n}}$, which is perpendicular to the plane containing **A** and **B** and defined by the right-hand rule.

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then use of Eqs. (3.25) and (3.26) leads to

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) \\ &\quad + \hat{\mathbf{z}}(A_xB_y - A_yB_x). \qquad (3.27)\end{aligned}$$

The cyclical form of the result given by Eq. (3.27) allows us to express the cross product in the form of a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \qquad (3.28)$$

Example 3-1: Vectors and Angles

In Cartesian coordinates, vector $\mathbf{A}$ points from the origin to point $P_1 = (2, 3, 3)$, and vector $\mathbf{B}$ is directed from P_1 to point $P_2 = (1, -2, 2)$. Find:

(a) vector $\mathbf{A}$, its magnitude A, and unit vector $\hat{\mathbf{a}}$,
(b) the angle between $\mathbf{A}$ and the y axis,
(c) vector $\mathbf{B}$,
(d) the angle θ_{AB} between $\mathbf{A}$ and $\mathbf{B}$, and
(e) perpendicular distance from the origin to vector $\mathbf{B}$.

Solution: **(a)** Vector $\mathbf{A}$ is given by the position vector of $P_1 = (2, 3, 3)$ (Fig. 3-7). Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

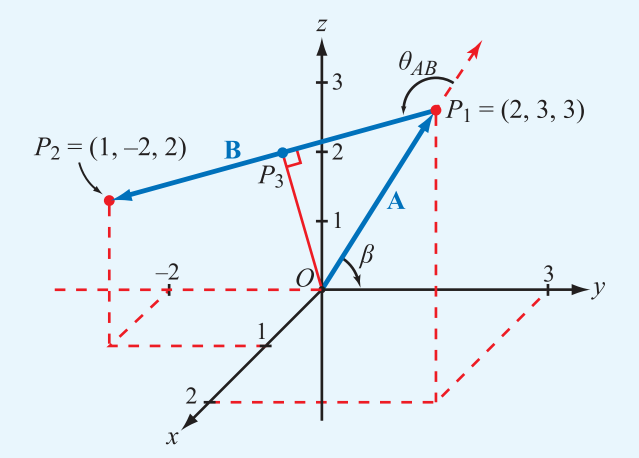


Figure 3-7 Geometry of Example 3-1.

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22},$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}.$$

(b) The angle β between $\mathbf{A}$ and the y axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ.$$

(c)

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\sqrt{27}}\right] = 145.1^\circ.$$

(e) The perpendicular distance between the origin and vector $\mathbf{B}$ is the distance $|\overrightarrow{OP_3}|$ shown in Fig. 3-7. From right triangle OP_1P_3,

$$\begin{aligned}|\overrightarrow{OP_3}| &= |\mathbf{A}| \sin(180^\circ - \theta_{AB}) \\ &= \sqrt{22}\sin(180^\circ - 145.1^\circ) = 2.68.\end{aligned}$$

Example 3-2: Cross Product

Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, compute (a) $\mathbf{A} \times \mathbf{B}$, (b) $\hat{\mathbf{y}} \times \mathbf{B}$, and (c) $(\hat{\mathbf{y}} \times \mathbf{B}) \cdot \mathbf{A}$.

Solution: **(a)** Application of Eq. (3.28) gives

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & -1 & 3 \\ 0 & 2 & -3 \end{vmatrix} \\ &= \hat{\mathbf{x}}((-1)\times(-3) - 3\times 2) - \hat{\mathbf{y}}(2\times(-3) - 3\times 0) \\ &\quad + \hat{\mathbf{z}}(2\times 2 - (-1\times 0)) \\ &= -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}6 + \hat{\mathbf{z}}4.\end{aligned}$$

(b) $\hat{\mathbf{y}} \times \mathbf{B} = \hat{\mathbf{y}} \times (\hat{\mathbf{y}}2 - \hat{\mathbf{z}}3) = -\hat{\mathbf{x}}3.$

(c) $(\hat{\mathbf{y}} \times \mathbf{B}) \cdot \mathbf{A} = -\hat{\mathbf{x}}3 \cdot (\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) = -6.$

Exercise 3-1: Find the distance vector between $P_1 = (1, 2, 3)$ and $P_2 = (-1, -2, 3)$ in Cartesian coordinates.

Answer: $\overrightarrow{P_1P_2} = -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$. (See EM.)

Exercise 3-2: Find the angle θ_{AB} between vectors $\mathbf{A}$ and $\mathbf{B}$ of Example 3-1 from the cross product between them.

Answer: $\theta_{AB} = 145.1°$. (See EM.)

Exercise 3-3: Find the angle between vector $\mathbf{B}$ of Example 3-1 and the z axis.

Answer: 101.1°. (See EM.)

Exercise 3-4: Vectors $\mathbf{A}$ and $\mathbf{B}$ lie in the y–z plane and both have the same magnitude of 2 (**Fig. E3.4**). Determine (a) $\mathbf{A} \cdot \mathbf{B}$ and (b) $\mathbf{A} \times \mathbf{B}$.

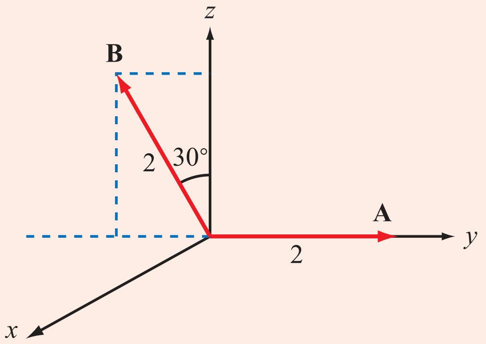


Figure E3.4

Answer: (a) $\mathbf{A} \cdot \mathbf{B} = -2$; (b) $\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}3.46$. (See EM.)

Exercise 3-5: If $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$, does it follow that $\mathbf{B} = \mathbf{C}$?

Answer: No. (See EM.)

3-1.5 Scalar and Vector Triple Products

When three vectors are multiplied, not all combinations of dot and cross products are meaningful. For example, the product

$$\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$$

does not make sense because $\mathbf{B} \cdot \mathbf{C}$ is a scalar, and the cross product of the vector $\mathbf{A}$ with a scalar is not defined under the rules of vector algebra. Other than the product of the form $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$, the only two meaningful products of three vectors are the scalar triple product and the vector triple product.

Scalar Triple Product

The dot product of a vector with the cross product of two other vectors is called a scalar triple product, so named because the result is a scalar. A scalar triple product obeys the cyclic order:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \quad (3.29)$$

The equalities hold as long as the cyclic order ($ABCABC\ldots$) is preserved. The scalar triple product of vectors $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{B} = (B_x, B_y, B_z)$, and $\mathbf{C} = (C_x, C_y, C_z)$ can be expressed in the form of a 3×3 determinant:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \quad (3.30)$$

The validity of Eqs. (3.29) and (3.30) can be verified by expanding $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in component form and carrying out the multiplications.

Vector Triple Product

The vector triple product involves the cross product of a vector with the cross product of two others, such as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \quad (3.31)$$

Since each cross product yields a vector, the result of a vector triple product is also a vector. The vector triple product does not obey the associative law. That is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}, \quad (3.32)$$

which means that it is important to specify which cross multiplication is to be performed first. By expanding the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in component form, it can be shown that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \quad (3.33)$$

which is known as the "bac-cab" rule.

Example 3-3: Vector Triple Product

Given $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}}2 + \hat{\mathbf{z}}3$, find $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and compare it with $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Solution:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

and

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{\mathbf{x}}3+\hat{\mathbf{y}}7-\hat{\mathbf{z}}2.$$

A similar procedure gives $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}2+\hat{\mathbf{y}}4+\hat{\mathbf{z}}$. The fact that the results of two vector triple products are different demonstrates the inequality stated in Eq. (3.32).

Concept Question 3-1: When are two vectors equal and when are they identical?

Concept Question 3-2: When is the position vector of a point identical to the distance vector between two points?

Concept Question 3-3: If $\mathbf{A}\cdot\mathbf{B}=0$, what is θ_{AB}?

Concept Question 3-4: If $\mathbf{A}\times\mathbf{B}=0$, what is θ_{AB}?

Concept Question 3-5: Is $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ a vector triple product?

Concept Question 3-6: If $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{C}$, does it follow that $\mathbf{B}=\mathbf{C}$?

3-2 Orthogonal Coordinate Systems

A three-dimensional coordinate system allows us to uniquely specify locations of points in space and the magnitudes and directions of vectors. Coordinate systems may be orthogonal or nonorthogonal.

▶ An ***orthogonal coordinate system*** is one in which coordinates are measured along locally mutually perpendicular axes. ◀

Nonorthogonal systems are very specialized and seldom used in solving practical problems. Many orthogonal coordinate systems have been devised, but the most commonly used are

- the Cartesian (also called rectangular),
- the cylindrical, and
- the spherical coordinate system.

Why do we need more than one coordinate system? Whereas a point in space has the same location and an object has the same shape regardless of which coordinate system is used to describe them, the solution of a practical problem can be greatly facilitated by the choice of a coordinate system that best fits the geometry under consideration. The following subsections examine the properties of each of the aforementioned orthogonal systems, and Section 3-3 describes how a point or vector may be transformed from one system to another.

3-2.1 Cartesian Coordinates

The Cartesian coordinate system was introduced in Section 3-1 to illustrate the laws of vector algebra. Instead of repeating these laws for the Cartesian system, we summarize them in **Table 3-1**. Differential calculus involves the use of differential lengths, areas, and volumes. In Cartesian coordinates, a ***differential length vector*** (**Fig. 3-8**) is expressed as

$$d\mathbf{l} = \hat{\mathbf{x}}\,dl_x+\hat{\mathbf{y}}\,dl_y+\hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}\,dx+\hat{\mathbf{y}}\,dy+\hat{\mathbf{z}}\,dz, \qquad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.

A ***differential area vector*** $d\mathbf{s}$ is a vector with magnitude ds equal to the product of two differential lengths (such as dl_y and dl_z) and direction specified by a unit vector along the third

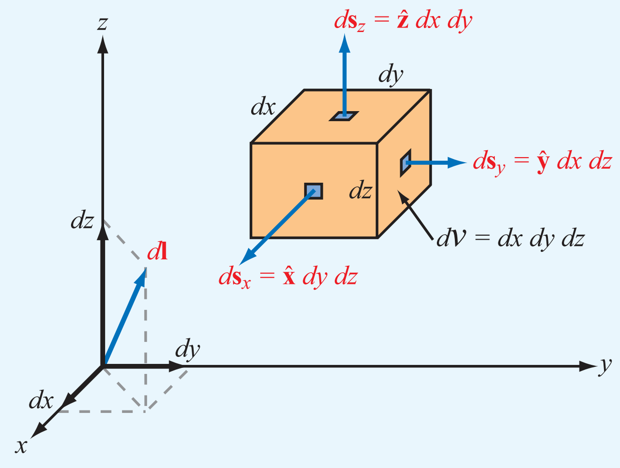


Figure 3-8 Differential length, area, and volume in Cartesian coordinates.

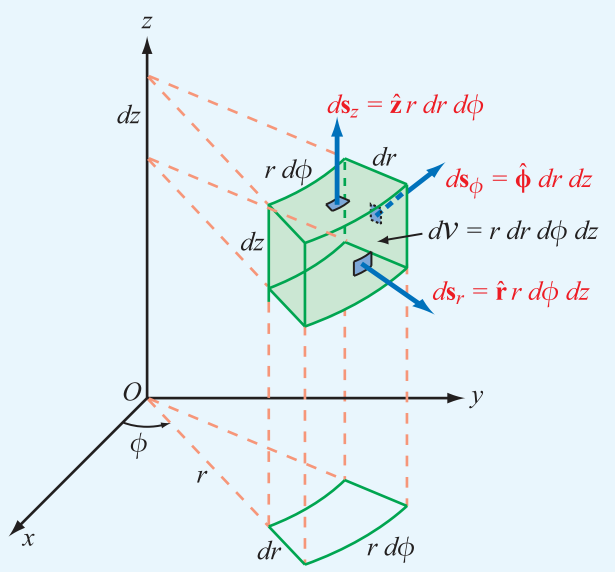


Figure 3-10 Differential areas and volume in cylindrical coordinates.

The differential volume is the product of the three differential lengths,

$$d\mathcal{V} = dl_r\, dl_\phi\, dl_z = r\, dr\, d\phi\, dz. \tag{3.44}$$

These properties of the cylindrical coordinate system are summarized in **Table 3-1**.

Example 3-4: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector $\mathbf{A}$ shown in **Fig. 3-11** in cylindrical coordinates.

Solution: In triangle OP_1P_2,

$$\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.$$

Hence,

$$\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{\mathbf{r}} r_0 - \hat{\mathbf{z}} h,$$

and

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{r}} r_0 - \hat{\mathbf{z}} h}{\sqrt{r_0^2 + h^2}}.$$

We note that the expression for $\mathbf{A}$ is independent of ϕ_0. This implies that all vectors from point P_1 to any point on the circle defined by $r = r_0$ in the x–y plane are equal in the cylindrical

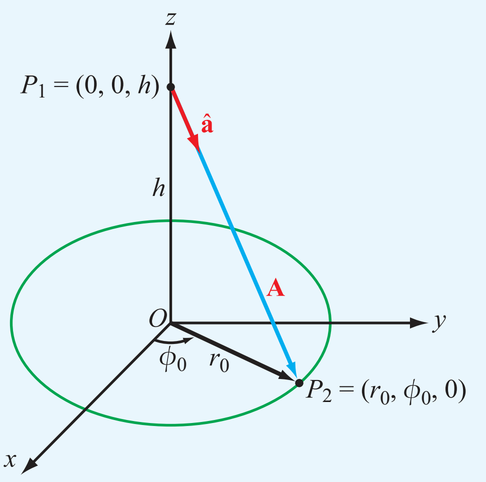


Figure 3-11 Geometry of Example 3-4.

coordinate system, which is not true. The ambiguity can be resolved by specifying that $\mathbf{A}$ passes through a point whose $\phi = \phi_0$.

Example 3-5: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30^\circ \le \phi \le 60^\circ$, and $0 \le z \le 3$ (**Fig. 3-12**).

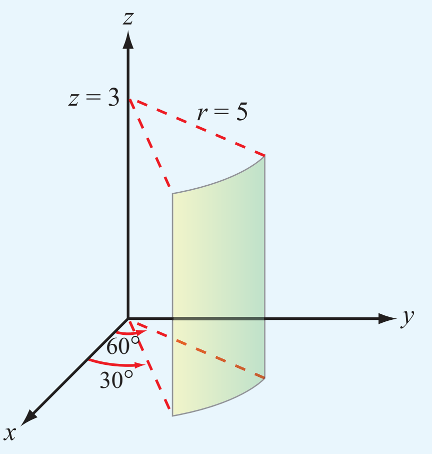


Figure 3-12 Cylindrical surface of Example 3-5.

Solution: The prescribed surface is shown in **Fig. 3-12**. Use of Eq. (3.43a) for a surface element with constant r gives

$$S = r\int_{\phi=30^\circ}^{60^\circ} d\phi \int_{z=0}^{3} dz = 5\phi\Big|_{\pi/6}^{\pi/3}\, z\Big|_0^3 = \frac{5\pi}{2}.$$

Note that ϕ had to be converted to radians before evaluating the integration limits.

Exercise 3-6: A circular cylinder of radius $r = 5$ cm is concentric with the z axis and extends between $z = -3$ cm and $z = 3$ cm. Use Eq. (3.44) to find the cylinder's volume.

Answer: 471.2 cm^3. (See EM.)

3-2.3 Spherical Coordinates

In the spherical coordinate system, the location of a point in space is uniquely specified by the variables R, θ, and ϕ (**Fig. 3-13**). The range coordinate R, which measures the distance from the origin to the point, describes a sphere of radius R centered at the origin. The ***zenith angle*** θ is measured from the positive z axis and it describes a conical surface with its apex at the origin, and the azimuth angle ϕ is the same as in cylindrical coordinates. The ranges of R, θ, and ϕ are $0 \le R < \infty$, $0 \le \theta \le \pi$, and $0 \le \phi < 2\pi$. The base vectors $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ obey the right-hand cyclic relations:

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}. \tag{3.45}$$

A vector with components A_R, A_θ, and A_ϕ is written as

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi, \tag{3.46}$$

and its magnitude is

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A}\cdot\mathbf{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}. \tag{3.47}$$

The position vector of point $P(R_1, \theta_1, \phi_1)$ is simply

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{R}}R_1, \tag{3.48}$$

while keeping in mind that $\hat{\mathbf{R}}$ is implicitly dependent on θ_1 and ϕ_1.

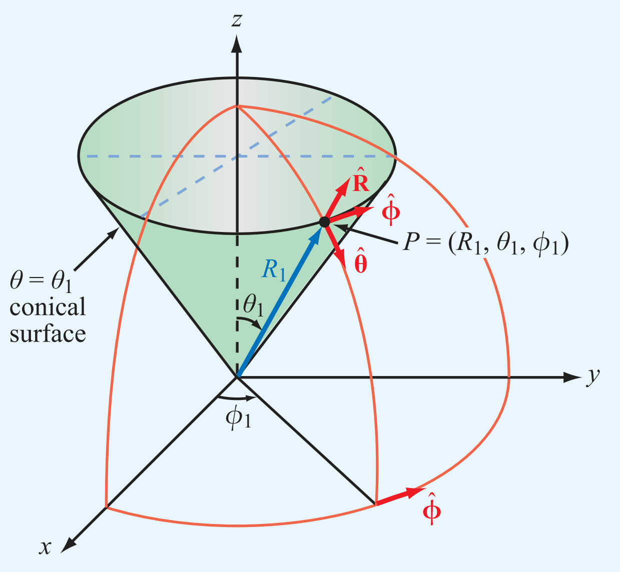


Figure 3-13 Point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates.

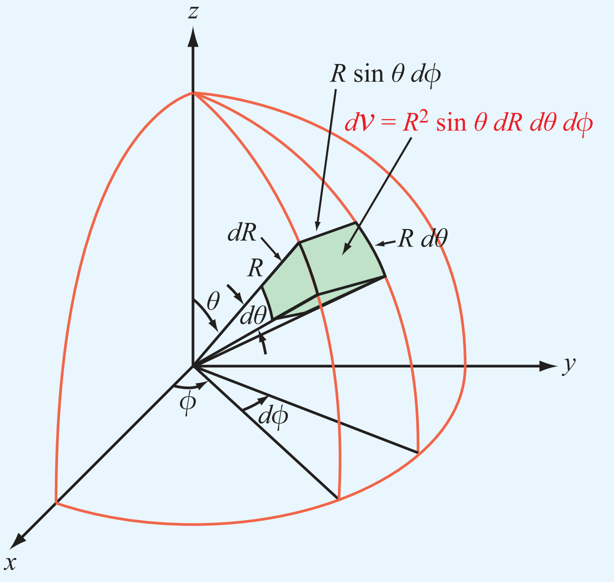


Figure 3-14 Differential volume in spherical coordinates.

As shown in Fig. 3-14, the differential lengths along $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are

$$dl_R = dR, \quad dl_\theta = R\,d\theta, \quad dl_\phi = R\sin\theta\,d\phi. \qquad (3.49)$$

Hence, the expressions for the vector differential length $d\mathbf{l}$, the vector differential surface $d\mathbf{s}$, and the differential volume $d\upsilon$ are

$$d\mathbf{l} = \hat{\mathbf{R}}\,dl_R + \hat{\boldsymbol{\theta}}\,dl_\theta + \hat{\boldsymbol{\phi}}\,dl_\phi$$

$$= \hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi, \qquad (3.50a)$$

$$d\mathbf{s}_R = \hat{\mathbf{R}}\,dl_\theta\,dl_\phi = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi \qquad (3.50b)$$

(θ–ϕ spherical surface),

$$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,dl_R\,dl_\phi = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi \qquad (3.50c)$$

(R–ϕ conical surface),

$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dl_R\,dl_\theta = \hat{\boldsymbol{\phi}}R\,dR\,d\theta \quad (R\text{–}\theta \text{ plane}), \qquad (3.50d)$$

$$d\upsilon = dl_R\,dl_\theta\,dl_\phi = R^2\sin\theta\,dR\,d\theta\,d\phi. \qquad (3.50e)$$

Example 3-6: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

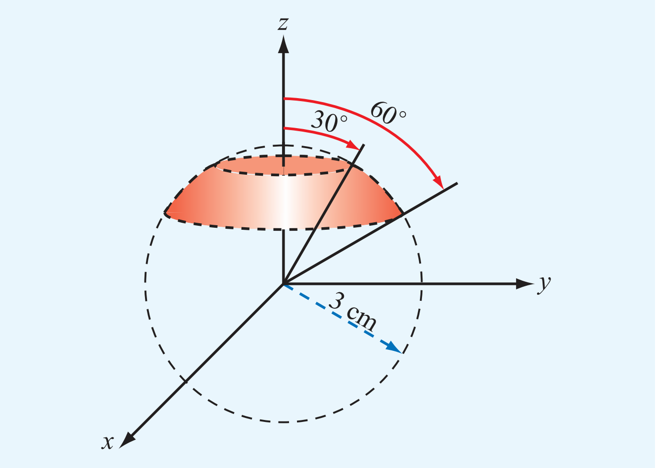


Figure 3-15 Spherical strip of Example 3-6.

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius R gives

$$S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin\theta\,d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 9(-\cos\theta)\Big|_{30^\circ}^{60^\circ}\ \phi\Big|_0^{2\pi} \qquad (\text{cm}^2)$$

$$= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7\ \text{cm}^2.$$

Example 3-7: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density ρ_v given by

$$\rho_v = 4\cos^2\theta \qquad (\text{C/m}^3).$$

Find the total charge Q contained in the sphere.

Solution:

$$Q = \int_\upsilon \rho_v\,d\upsilon$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=0}^{2\times10^{-2}} (4\cos^2\theta)R^2\sin\theta\,dR\,d\theta\,d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi}\left(\frac{R^3}{3}\right)\Big|_0^{2\times10^{-2}} \sin\theta\cos^2\theta\,d\theta\,d\phi$$

$$= \frac{32}{3}\times10^{-6}\int_0^{2\pi}\left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} d\phi$$

$$= \frac{64}{9}\times10^{-6}\int_0^{2\pi} d\phi$$

$$= \frac{128\pi}{9}\times10^{-6} = 44.68 \qquad (\mu\text{C}).$$

Note that the limits on R were converted to meters prior to evaluating the integral on R.

section, we shall establish the relations between the variables $x, y, z)$ of the Cartesian system, (r, ϕ, z) of the cylindrical system, and (R, θ, ϕ) of the spherical system. These relations will then be used to transform expressions for vectors expressed in any one of the three systems into expressions applicable in the other two.

Example 3-10: Vector Component

At a given point in space, vectors **A** and **B** are given in cylindrical coordinates by

$$\mathbf{A} = \hat{\mathbf{r}}2 + \hat{\boldsymbol{\phi}}3 - \hat{\mathbf{z}},$$
$$\mathbf{B} = \hat{\mathbf{r}} + \hat{\mathbf{z}}.$$

Determine (a) the scalar component of **B**, or projection, in the direction of **A**, (b) the vector component of **B** in the direction of **A**, and (c) the vector component of **B** perpendicular to **A**.

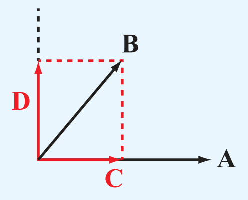


Figure 3-19 Vectors **A**, **B**, **C**, and **D** of Example 3-10.

Solution: **(a)** Let us denote the scalar component of **B** in the direction of **A** as C, as shown in **Fig. 3-19**. Thus,

$$C = \mathbf{B} \cdot \hat{\mathbf{a}} = \mathbf{B} \cdot \frac{\mathbf{A}}{|\mathbf{A}|} = (\hat{\mathbf{r}} + \hat{\mathbf{z}}) \cdot \frac{(\hat{\mathbf{r}}2 + \hat{\boldsymbol{\phi}}3 - \hat{\mathbf{z}})}{\sqrt{4+9+1}} = \frac{2-1}{\sqrt{14}} = 0.267.$$

(b) The vector component of **B** in the direction of **A** is given by the product of the scalar component C and the unit vector $\hat{\mathbf{a}}$:

$$\mathbf{C} = \hat{\mathbf{a}}C = \frac{\mathbf{A}}{|\mathbf{A}|}\,C = \frac{(\hat{\mathbf{r}}2 + \hat{\boldsymbol{\phi}}3 - \hat{\mathbf{z}})}{\sqrt{14}} \times 0.267$$
$$= \hat{\mathbf{r}}0.143 + \hat{\boldsymbol{\phi}}0.214 - \hat{\mathbf{z}}0.071.$$

(c) The vector component of **B** perpendicular to **A** is equal to **B** minus **C**:

$$\mathbf{D} = \mathbf{B} - \mathbf{C} = (\hat{\mathbf{r}} + \hat{\mathbf{z}}) - (\hat{\mathbf{r}}0.143 + \hat{\boldsymbol{\phi}}0.214 - \hat{\mathbf{z}}0.071)$$
$$= \hat{\mathbf{r}}0.857 - \hat{\boldsymbol{\phi}}0.214 + \hat{\mathbf{z}}0.929.$$

Concept Question 3-7: Why do we use more than one coordinate system?

Concept Question 3-8: Why is it that the base vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ are independent of the location of a point, but $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are not?

Concept Question 3-9: What are the cyclic relations for the base vectors in (a) Cartesian coordinates, (b) cylindrical coordinates, and (c) spherical coordinates?

Concept Question 3-10: How is the position vector of a point in cylindrical coordinates related to its position vector in spherical coordinates?

Exercise 3-7: Point $P = (2\sqrt{3}, \pi/3, -2)$ is given in cylindrical coordinates. Express P in spherical coordinates.

Answer: $P = (4, 2\pi/3, \pi/3)$. (See EM.)

Exercise 3-8: Transform vector

$$\mathbf{A} = \hat{\mathbf{x}}(x+y) + \hat{\mathbf{y}}(y-x) + \hat{\mathbf{z}}z$$

from Cartesian to cylindrical coordinates.

Answer: $\mathbf{A} = \hat{\mathbf{r}}r - \hat{\boldsymbol{\phi}}r + \hat{\mathbf{z}}z$. (See EM.)

3-4 Gradient of a Scalar Field

When dealing with a scalar physical quantity whose magnitude depends on a single variable, such as the temperature T as a function of height z, the rate of change of T with height can be described by the derivative dT/dz. However, if T is also a function of x and y, its spatial rate of change becomes more difficult to describe because we now have to deal with three separate variables. The differential change in T along x, y, and z can be described in terms of the partial derivatives of T with respect to the three coordinate variables, but it is not immediately obvious as to how we should combine the three partial derivatives so as to describe the spatial rate of change of T along a specified direction. Furthermore, many of the quantities we deal with in electromagnetics are vectors; therefore, both their magnitudes and directions may vary with spatial position. To this end, we introduce three fundamental operators to describe the differential spatial variations of scalars and vectors: the ***gradient***, ***divergence***, and ***curl*** operators. The gradient operator applies to scalar fields and is the subject of the present section. The other two operators, which apply to vector fields, are discussed in succeeding sections.

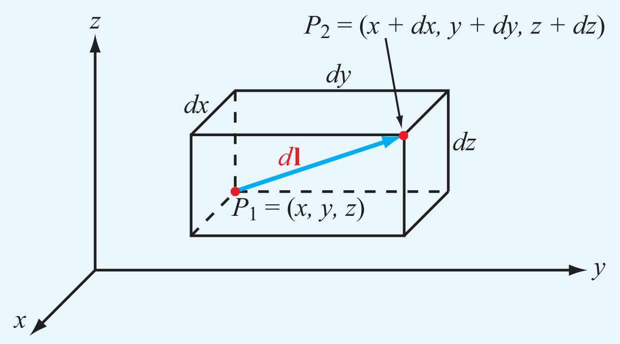


Figure 3-20 Differential distance vector $d\mathbf{l}$ between points P_1 and P_2.

Suppose that $T_1 = T(x,y,z)$ is the temperature at point $P_1 = (x,y,z)$ in some region of space, and

$$T_2 = T(x+dx,\, y+dy,\, z+dz)$$

is the temperature at a nearby point $P_2 = (x+dx,\, y+dy,\, z+dz)$ (**Fig. 3-20**). The differential distances dx, dy, and dz are the components of the differential distance vector $d\mathbf{l}$. That is,

$$d\mathbf{l} = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz. \qquad (3.69)$$

From differential calculus, the temperature difference between points P_1 and P_2, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz. \qquad (3.70)$$

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$\begin{aligned} dT &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l} \\ &= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l}. \qquad (3.71) \end{aligned}$$

The vector inside the square brackets in Eq. (3.71) relates the change in temperature dT to a vector change in direction $d\mathbf{l}$. This vector is called the ***gradient*** of T (or ***grad*** T for short) and denoted ∇T:

$$\nabla T = \text{grad}\ T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\,. \qquad (3.72)$$

Equation (3.71) then can be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \qquad (3.73)$$

The symbol ∇ is called the ***del*** or ***gradient operator*** and is defined as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \textbf{(Cartesian)}. \qquad (3.74)$$

► Whereas the gradient operator itself has no physical meaning, it attains a physical meaning once it operates on a scalar quantity, and the result of the operation is a vector with magnitude equal to the maximum rate of change of the physical quantity per unit distance and pointing in the direction of maximum increase. ◄

With $d\mathbf{l} = \hat{\mathbf{a}}_l\,dl$, where $\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$, the ***directional derivative*** of T along $\hat{\mathbf{a}}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \qquad (3.75)$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1,y_1,z_1)$ and $T_2 = T(x_2,y_2,z_2)$ are the values of T at points

$$P_1 = (x_1,y_1,z_1) \qquad \text{and} \qquad P_2 = (x_2,y_2,z_2)$$

not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}. \qquad (3.76)$$

Example 3-11: Directional Derivative

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at $(1,-1,2)$.

Solution: First, we find the gradient of T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2 + y^2 z) = \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

We denote $\mathbf{l}$ as the given direction,

$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.$$

Its unit vector is

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.$$

Application of Eq. (3.75) gives

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}} \right)$$
$$= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$

At $(1, -1, 2)$,

$$\left. \frac{dT}{dl} \right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.$$

3-4.1 Gradient Operator in Cylindrical and Spherical Coordinates

Even though Eq. (3.73) was derived using Cartesian coordinates, it should have counterparts in other coordinate systems. To convert Eq. (3.72) into cylindrical coordinates (r, ϕ, z), we start by restating the coordinate relations

$$r = \sqrt{x^2 + y^2}, \qquad \tan\phi = \frac{y}{x}. \tag{3.77}$$

From differential calculus,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial x}. \tag{3.78}$$

Since z is orthogonal to x and $\partial z/\partial x = 0$, the last term in Eq. (3.78) vanishes. From the coordinate relations given by Eq. (3.77), it follows that

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\phi, \tag{3.79a}$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{r}\sin\phi. \tag{3.79b}$$

Hence,

$$\frac{\partial T}{\partial x} = \cos\phi \frac{\partial T}{\partial r} - \frac{\sin\phi}{r}\frac{\partial T}{\partial \phi}. \tag{3.80}$$

This expression can be used to replace the coefficient of $\hat{\mathbf{x}}$ in Eq. (3.72), and a similar procedure can be followed to obtain an expression for $\partial T/\partial y$ in terms of r and ϕ. If, in addition, we use the relations $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ and $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ [from Eqs. (3.57a) and (3.57b)], then Eq. (3.72) becomes

$$\nabla T = \hat{\mathbf{r}}\frac{\partial T}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial T}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}. \tag{3.81}$$

Hence, the gradient operator in cylindrical coordinates can be expressed as

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \qquad \textbf{(cylindrical)} \tag{3.82}$$

A similar procedure leads to the expression for the gradient in spherical coordinates:

$$\nabla = \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}. \tag{3.83}$$

(spherical)

3-4.2 Properties of the Gradient Operator

For any two scalar functions U and V, the following relations apply:

(1) $\nabla(U + V) = \nabla U + \nabla V$, (3.84a)

(2) $\nabla(UV) = U\,\nabla V + V\,\nabla U$, (3.84b)

(3) $\nabla V^n = nV^{n-1}\,\nabla V$, for any n. (3.84c)

Example 3-12: Calculating the Gradient

Find the gradient of each of the following scalar functions and then evaluate it at the given point.

(a) $V_1 = 24V_0\cos(\pi y/3)\sin(2\pi z/3)$ at $(3, 2, 1)$ in Cartesian coordinates,

(b) $V_2 = V_0 e^{-2r}\sin 3\phi$ at $(1, \pi/2, 3)$ in cylindrical coordinates,

(c) $V_3 = V_0\,(a/R)\cos 2\theta$ at $(2a, 0, \pi)$ in spherical coordinates.

Solution: (a) Using Eq. (3.72) for ∇,

$$\nabla V_1 = \hat{\mathbf{x}}\frac{\partial V_1}{\partial x} + \hat{\mathbf{y}}\frac{\partial V_1}{\partial y} + \hat{\mathbf{z}}\frac{\partial V_1}{\partial z}$$
$$= -\hat{\mathbf{y}}8\pi V_0 \sin\frac{\pi y}{3}\sin\frac{2\pi z}{3} + \hat{\mathbf{z}}16\pi V_0\cos\frac{\pi y}{3}\cos\frac{2\pi z}{3}$$
$$= 8\pi V_0\left[-\hat{\mathbf{y}}\sin\frac{\pi y}{3}\sin\frac{2\pi z}{3} + \hat{\mathbf{z}}2\cos\frac{\pi y}{3}\cos\frac{2\pi z}{3}\right].$$

At $(3, 2, 1)$,

$$\nabla V_1 = 8\pi V_0\left[-\hat{\mathbf{y}}\sin^2\frac{2\pi}{3} + \hat{\mathbf{z}}2\cos^2\frac{2\pi}{3}\right] = \pi V_0\,[-\hat{\mathbf{y}}6 + \hat{\mathbf{z}}4].$$

(b) The function V_2 is expressed in terms of cylindrical variables. Hence, we need to use Eq. (3.82) for ∇:

$$\nabla V_2 = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) V_0 e^{-2r}\sin 3\phi$$
$$= -\hat{\mathbf{r}}2V_0 e^{-2r}\sin 3\phi + \hat{\boldsymbol{\phi}}(3V_0 e^{-2r}\cos 3\phi)/r$$
$$= \left[-\hat{\mathbf{r}}2\sin 3\phi + \hat{\boldsymbol{\phi}}\frac{3\cos 3\phi}{r}\right] V_0 e^{-2r}.$$

At $(1,\pi/2,3)$, $r = 1$ and $\phi = \pi/2$. Hence,

$$\nabla V_2 = \left[-\hat{\mathbf{r}}2\sin\frac{3\pi}{2} + \hat{\boldsymbol{\phi}}3\cos\frac{3\pi}{2}\right] V_0 e^{-2} = \hat{\mathbf{r}}2V_0 e^{-2} = \hat{\mathbf{r}}0.27V_0.$$

(c) As V_3 is expressed in spherical coordinates, we apply Eq. (3.83) to V_3:

$$\nabla V_3 = \left(\hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\right) V_0\left(\frac{a}{R}\right)\cos 2\theta$$
$$= -\hat{\mathbf{R}}\frac{V_0 a}{R^2}\cos 2\theta - \hat{\boldsymbol{\theta}}\frac{2V_0 a}{R^2}\sin 2\theta$$
$$= -[\hat{\mathbf{R}}\cos 2\theta + \hat{\boldsymbol{\theta}}2\sin 2\theta]\frac{V_0 a}{R^2}.$$

At $(2a,0,\pi)$, $R = 2a$ and $\theta = 0$, which yields

$$\nabla V_3 = -\hat{\mathbf{R}}\frac{V_0}{4a}.$$

3-5 Divergence of a Vector Field

From our brief introduction of Coulomb's law in Chapter 1, we know that an isolated, positive point charge q induces an electric field $\mathbf{E}$ in the space around it with the direction of $\mathbf{E}$ being outward away from the charge. Also, the strength (magnitude) of $\mathbf{E}$ is proportional to q and decreases with distance R from the charge as $1/R^2$. In a graphical presentation, a vector field is usually represented by ***field lines***, as shown in **Fig. 3-21**. The arrowhead denotes the direction of the field at the point where the field line is drawn, and the length of the line provides a qualitative depiction of the field's magnitude.

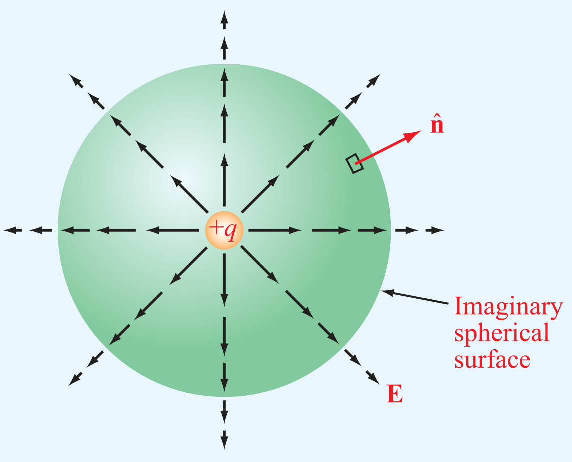


Figure 3-21 Flux lines of the electric field **E** due to a positive charge q.

At a surface boundary, ***flux density*** is defined as the amount of outward flux crossing a unit surface ds:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E}\cdot d\mathbf{s}}{|d\mathbf{s}|} = \frac{\mathbf{E}\cdot\hat{\mathbf{n}}\,ds}{ds} = \mathbf{E}\cdot\hat{\mathbf{n}}, \qquad (3.85)$$

where $\hat{\mathbf{n}}$ is the normal to $d\mathbf{s}$. The ***total flux*** outwardly crossing a closed surface S, such as the enclosed surface of the imaginary sphere outlined in **Fig. 3-21**, is

$$\text{Total flux} = \oint_S \mathbf{E}\cdot d\mathbf{s}. \qquad (3.86)$$

Let us now consider the case of a differential rectangular parallelepiped, such as a cube, whose edges align with the Cartesian axes shown in **Fig. 3-22**. The edges are of lengths Δx along x, Δy along y, and Δz along z. A vector field $\mathbf{E}(x,y,z)$ exists in the region of space containing the parallelepiped, and we wish to determine the flux of $\mathbf{E}$ through its total surface S. Since S includes six faces, we need to sum up the fluxes through all of them, and by definition, the flux through any face is the *outward* flux from the volume $\Delta \upsilon$ through that face.

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z = (\text{div}\,\mathbf{E})\,\Delta \upsilon, \qquad (3.93)$$

where $\Delta \upsilon = \Delta x\,\Delta y\,\Delta z$ and div **E** is a scalar function called the ***divergence*** of **E**, specified in Cartesian coordinates as

$$\text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \qquad (3.94)$$

$$\text{div}\,\mathbf{E} \triangleq \lim_{\Delta \upsilon\to 0}\frac{\oint_S \mathbf{E}\cdot d\mathbf{s}}{\Delta \upsilon}, \qquad (3.95)$$

where S encloses the elemental volume $\Delta \upsilon$. Instead of denoting the divergence of **E** by div **E**, it is common practice to denote it as $\nabla\cdot\mathbf{E}$. That is,

$$\nabla\cdot\mathbf{E} = \text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.96)$$

The divergence is a differential operator, it always operates on vectors, and the result of its operation is a scalar. This is in contrast with the gradient operator, which always operates on scalars and results in a vector. Expressions for the divergence of a vector in cylindrical and spherical coordinates are provided in Appendix C.

The divergence operator is distributive. That is, for any pair of vectors $\mathbf{E}_1$ and $\mathbf{E}_2$,

$$\nabla \cdot (\mathbf{E}_1 + \mathbf{E}_2) = \nabla \cdot \mathbf{E}_1 + \nabla \cdot \mathbf{E}_2. \tag{3.97}$$

If $\nabla \cdot \mathbf{E} = 0$, the vector field $\mathbf{E}$ is called ***divergenceless***.

The result given by Eq. (3.93) for a differential volume $\Delta \upsilon$ can be extended to relate the volume integral of $\nabla \cdot \mathbf{E}$ over any volume υ to the flux of $\mathbf{E}$ through the closed surface S that bounds υ. That is,

$$\int_{\upsilon} \nabla \cdot \mathbf{E}\, d\upsilon = \oint_S \mathbf{E} \cdot d\mathbf{s}. \tag{3.98}$$

(divergence theorem)

This relationship, known as the ***divergence theorem***, is used extensively in electromagnetics.

Example 3-13: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2 z$ at $(2, -2, 0)$;
(b) $\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta / R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta / R^2)$ at $(a/2, 0, \pi)$.

Solution: (a)

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2 z) \\ &= 6x + 0 + x^2 = x^2 + 6x.\end{aligned}$$

At $(2, -2, 0)$, $\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16.$

(b) From the expression given in Appendix C for the divergence of a vector in spherical coordinates, it follows that

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(E_\theta \sin\theta) \\ &\quad + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial \phi} \\ &= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right) \\ &= 0 - \frac{2a^3\cos\theta}{R^3} = -\frac{2a^3\cos\theta}{R^3}.\end{aligned}$$

At $R = a/2$ and $\theta = 0$, $\nabla \cdot \mathbf{E}\Big|_{(a/2,0,\pi)} = -16.$

Exercise 3-13: Given $\mathbf{A} = e^{-2y}(\hat{\mathbf{x}}\sin 2x + \hat{\mathbf{y}}\cos 2x)$, find $\nabla \cdot \mathbf{A}$.

Answer: $\nabla \cdot \mathbf{A} = 0$. (See EM.)

Exercise 3-14: Given $\mathbf{A} = \hat{\mathbf{r}}r\cos\phi + \hat{\boldsymbol{\phi}}r\sin\phi + \hat{\mathbf{z}}3z$, find $\nabla \cdot \mathbf{A}$ at $(2, 0, 3)$.

Answer: $\nabla \cdot \mathbf{A} = 6$. (See EM.)

Exercise 3-15: If $\mathbf{E} = \hat{\mathbf{R}}AR$ in spherical coordinates, calculate the flux of $\mathbf{E}$ through a spherical surface of radius a, centered at the origin.

Answer: $\oint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi A a^3$. (See EM.)

Exercise 3-16: Verify the divergence theorem by calculating the volume integral of the divergence of the field $\mathbf{E}$ of Exercise 3.15 over the volume bounded by the surface of radius a.

Exercise 3-17: The arrow representation in **Fig. E3.17** represents the vector field $\mathbf{A} = \hat{\mathbf{x}}x - \hat{\mathbf{y}}y$. At a given point in space, $\mathbf{A}$ has a positive divergence $\nabla \cdot \mathbf{A}$ if the net flux flowing outward through the surface of an imaginary infinitesimal volume centered at that point is positive, $\nabla \cdot \mathbf{A}$ is negative if the net flux is into the volume, and $\nabla \cdot \mathbf{A} = 0$ if the same amount of flux enters into the volume as leaves it. Determine $\nabla \cdot \mathbf{A}$ everywhere in the x–y plane.

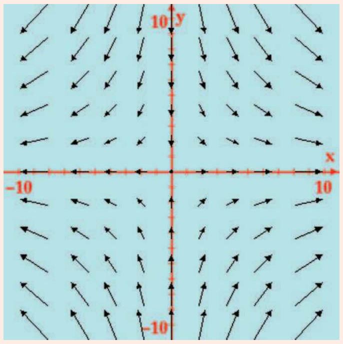


Figure E3.17

Answer: $\nabla \cdot \mathbf{A} = 0$ everywhere. (See EM.)

3-6 Curl of a Vector Field

So far we have defined and discussed two of the three fundamental operators used in vector analysis: the gradient of a scalar and the divergence of a vector. Now we introduce the ***curl operator***. The curl of a vector field $\mathbf{B}$ describes its rotational property, or ***circulation***. The circulation of $\mathbf{B}$ is defined as the line integral of $\mathbf{B}$ around a closed contour C;

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}. \tag{3.99}$$

To gain a physical understanding of this definition, we consider two examples. The first is for a uniform field $\mathbf{B} = \hat{\mathbf{x}}B_0$, whose field lines are as depicted in **Fig. 3-23(a)**. For the rectangular contour $abcd$ shown in the figure, we have

$$\begin{aligned}\text{Circulation} &= \int_a^b \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{x}}\,dx + \int_b^c \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{y}}\,dy \\ &\quad + \int_c^d \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{x}}\,dx + \int_d^a \hat{\mathbf{x}}B_0 \cdot \hat{\mathbf{y}}\,dy \\ &= B_0\,\Delta x - B_0\,\Delta x = 0,\end{aligned} \tag{3.100}$$

where $\Delta x = b - a = c - d$ and, because $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$, the second and fourth integrals are zero. According to Eq. (3.100), *the circulation of a uniform field is zero.*

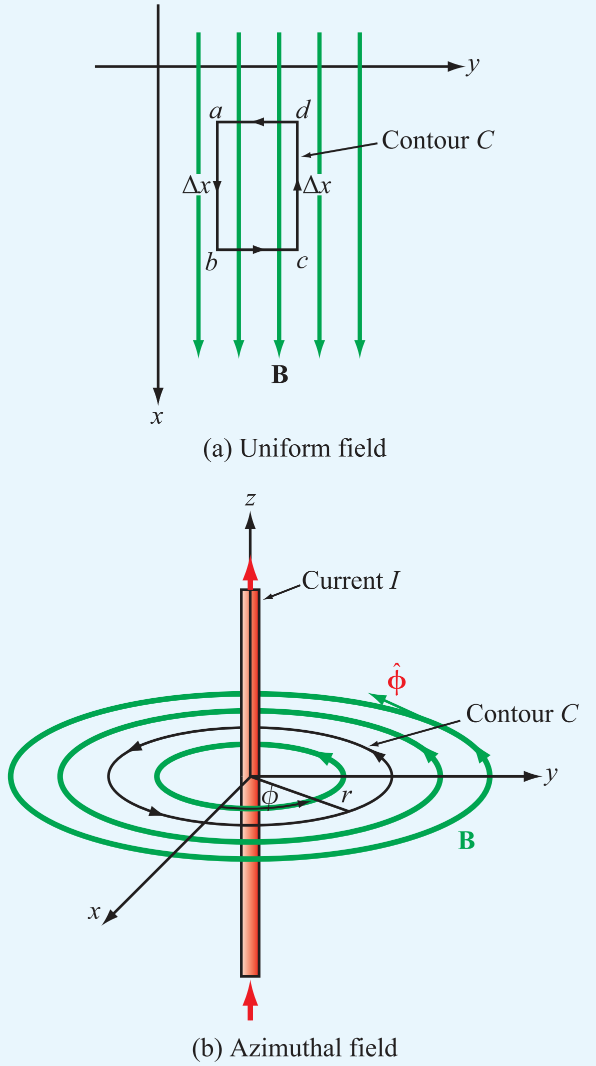


Figure 3-23 Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Next, we consider the magnetic flux density $\mathbf{B}$ induced by an infinite wire carrying a dc current I. If the current is in free space and it is oriented along the z direction, then from Eq. (1.13),

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I}{2\pi r}, \tag{3.101}$$

where μ_0 is the permeability of free space and r is the radial distance from the current in the x–y plane. The direction of $\mathbf{B}$ is along the azimuth unit vector $\hat{\boldsymbol{\phi}}$. The field lines of $\mathbf{B}$ are concentric circles around the current, as shown in **Fig. 3-23(b)**.

For a circular contour C of radius r centered at the origin in the x–y plane, the differential length vector $d\mathbf{l} = \hat{\boldsymbol{\phi}} r\, d\phi$, and the circulation of $\mathbf{B}$ is

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \cdot \hat{\boldsymbol{\phi}} r\, d\phi = \mu_0 I. \quad (3.102)$$

In this case, the circulation is not zero. However, had the contour C been in the x–z or y–z planes, $d\mathbf{l}$ would not have had a $\hat{\boldsymbol{\phi}}$ component, and the integral would have yielded a zero circulation. Clearly, the circulation of $\mathbf{B}$ depends on the choice of contour and the direction in which it is traversed. To describe the circulation of a tornado, for example, we would like to choose our contour such that the circulation of the wind field is maximum, and we would like the circulation to have both a magnitude and a direction with the direction being toward the tornado's vortex. The curl operator embodies these properties. The curl of a vector field $\mathbf{B}$, denoted curl $\mathbf{B}$ or $\nabla \times \mathbf{B}$, is defined as

$$\nabla \times \mathbf{B} = \text{curl } \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}. \quad (3.103)$$

▶ Curl $\mathbf{B}$ is the circulation of $\mathbf{B}$ per unit area, with the area Δs of the contour C being oriented such that the circulation is maximum. ◀

The direction of curl $\mathbf{B}$ is $\hat{\mathbf{n}}$, the unit normal of Δs, defined according to the right-hand rule with the four fingers of the right hand following the contour direction $d\mathbf{l}$ and the thumb pointing along $\hat{\mathbf{n}}$ (**Fig. 3-24**). When we use the notation $\nabla \times \mathbf{B}$ to denote curl $\mathbf{B}$, it should *not* be interpreted as the cross product of ∇ and $\mathbf{B}$.

For a vector $\mathbf{B}$ specified in Cartesian coordinates as

$$\mathbf{B} = \hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z, \quad (3.104)$$

it can be shown, through a rather lengthy derivation, that Eq. (3.103) leads to

$$\begin{aligned} \nabla \times \mathbf{B} &= \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \\ &\quad + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}. \end{aligned} \quad (3.105)$$

Expressions for $\nabla \times \mathbf{B}$ are given in Appendix C for the three orthogonal coordinate systems considered in this chapter.

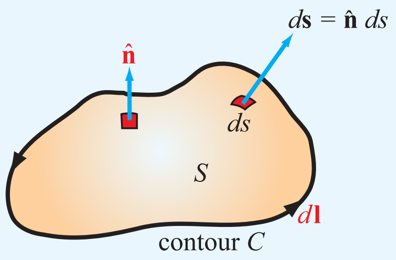


Figure 3-24 The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

3-6.1 Vector Identities Involving the Curl

For any two vectors $\mathbf{A}$ and $\mathbf{B}$ and scalar V,

$$(1) \quad \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}, \quad (3.106a)$$

$$(2) \quad \nabla \cdot (\nabla \times \mathbf{A}) = 0, \quad (3.106b)$$

$$(3) \quad \nabla \times (\nabla V) = 0. \quad (3.106c)$$

3-6.2 Stokes's Theorem

▶ ***Stokes's theorem*** converts the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S. ◀

For the geometry shown in **Fig. 3-24**, ***Stokes's theorem*** states

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}. \quad (3.107)$$

(Stokes's theorem)

Its validity follows from the definition of $\nabla \times \mathbf{B}$ given by Eq. (3.103). If $\nabla \times \mathbf{B} = 0$, the field $\mathbf{B}$ is said to be ***conservative*** or ***irrotational*** because its circulation, represented by the right-hand side of Eq. (3.107), is zero, irrespective of the contour chosen.

Example 3-14: Verification of Stokes's Theorem

For vector field $\mathbf{B} = \hat{\mathbf{z}} \cos\phi/r$, verify Stokes's theorem for a segment of a cylindrical surface defined by $r = 2$, $\pi/3 \leq \phi \leq \pi/2$, and $0 \leq z \leq 3$ (**Fig. 3-25**).

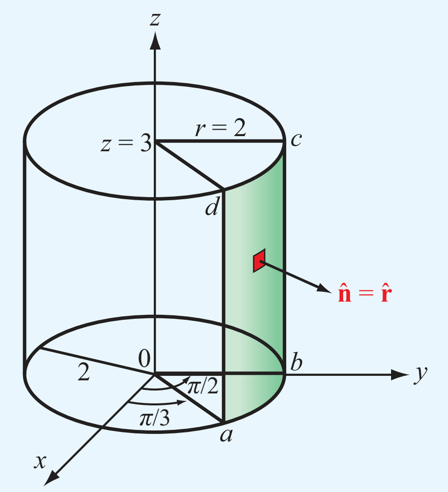


Figure 3-25 Geometry of Example 3-14.

Solution: Stokes's theorem states that

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$

Left-hand side: With $\mathbf{B}$ having only a component $B_z = \cos\phi/r$, use of the expression for $\nabla \times \mathbf{B}$ in cylindrical coordinates in Appendix C gives

$$\begin{aligned}
\nabla \times \mathbf{B} &= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \\
&\quad + \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r}(r B_\phi) - \frac{\partial B_r}{\partial \phi} \right) \\
&= \hat{\mathbf{r}} \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\cos\phi}{r} \right) - \hat{\boldsymbol{\phi}} \frac{\partial}{\partial r} \left(\frac{\cos\phi}{r} \right) \\
&= -\hat{\mathbf{r}} \frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}} \frac{\cos\phi}{r^2}.
\end{aligned}$$

The integral of $\nabla \times \mathbf{B}$ over the specified surface S is

$$\begin{aligned}
\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} &= \int_{z=0}^{3} \int_{\phi=\pi/3}^{\pi/2} \left(-\hat{\mathbf{r}} \frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}} \frac{\cos\phi}{r^2} \right) \cdot \hat{\mathbf{r}} r \, d\phi \, dz \\
&= \int_0^3 \int_{\pi/3}^{\pi/2} -\frac{\sin\phi}{r} \, d\phi \, dz = -\frac{3}{2r} = -\frac{3}{4}.
\end{aligned}$$

Right-hand side: The surface S is bounded by contour $C = abcd$ shown in **Fig. 3-25**. The direction of C is chosen so that it is compatible with the surface normal $\hat{\mathbf{r}}$ by the right-hand rule. Hence,

$$\begin{aligned}
\oint_C \mathbf{B} \cdot d\mathbf{l} &= \int_a^b \mathbf{B}_{ab} \cdot d\mathbf{l} + \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{l} \\
&\quad + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{l} + \int_d^a \mathbf{B}_{da} \cdot d\mathbf{l},
\end{aligned}$$

where $\mathbf{B}_{ab}$, $\mathbf{B}_{bc}$, $\mathbf{B}_{cd}$, and $\mathbf{B}_{da}$ are the field $\mathbf{B}$ along segments ab, bc, cd, and da, respectively. Over segment ab, the dot product of $\mathbf{B}_{ab} = \hat{\mathbf{z}}(\cos\phi)/2$ and $d\mathbf{l} = \hat{\boldsymbol{\phi}} r \, d\phi$ is zero, and the same is true for segment cd. Over segment bc, $\phi = \pi/2$; hence, $\mathbf{B}_{bc} = \hat{\mathbf{z}}(\cos\pi/2)/2 = 0$. For the last segment, $\mathbf{B}_{da} = \hat{\mathbf{z}}(\cos\pi/3)/2 = \hat{\mathbf{z}}/4$ and $d\mathbf{l} = \hat{\mathbf{z}} \, dz$. Hence,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_d^a \left(\hat{\mathbf{z}} \frac{1}{4} \right) \cdot \hat{\mathbf{z}} \, dz = \int_3^0 \frac{1}{4} \, dz = -\frac{3}{4},$$

which is the same as the result obtained by evaluating the left-hand side of Stokes's equation.

Exercise 3-18: Find $\nabla \times \mathbf{A}$ at $(2, 0, 3)$ in cylindrical coordinates for the vector field

$$\mathbf{A} = \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi.$$

Answer: (See EM.)

$$\nabla \times \mathbf{A} = \left(\hat{\mathbf{r}} \frac{10\cos\phi}{r} + \frac{\hat{\mathbf{z}} \, 10 e^{-2r}}{r} \sin\phi \right) \Bigg|_{(2,0,3)} = \hat{\mathbf{r}} 5.$$

Exercise 3-19: Find $\nabla \times \mathbf{A}$ at $(3, \pi/6, 0)$ in spherical coordinates for the vector field $\mathbf{A} = \hat{\boldsymbol{\theta}} 12 \sin\theta$.

Answer: (See EM.)

$$\nabla \times \mathbf{A} = \hat{\boldsymbol{\phi}} \frac{12 \sin\theta}{R} \Bigg|_{(3,\pi/6,0)} = \hat{\boldsymbol{\phi}} 2.$$

3-7 Laplacian Operator

In later chapters, we sometimes deal with problems involving multiple combinations of operations on scalars and vectors. A frequently encountered combination is the divergence of the gradient of a scalar. For a scalar function V defined in Cartesian coordinates, its gradient is

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z = \mathbf{A}, \tag{3.108}$$

where we defined a vector $\mathbf{A}$ with components $A_x = \partial V/\partial x$, $A_y = \partial V/\partial y$, and $A_z = \partial V/\partial z$. The divergence of ∇V is

$$\begin{aligned}\nabla\cdot(\nabla V) = \nabla\cdot\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.\end{aligned} \tag{3.109}$$

For convenience, $\nabla\cdot(\nabla V)$ is called the ***Laplacian*** of V and is denoted by $\nabla^2 V$ (the symbol ∇^2 is pronounced "del square"). That is,

$$\nabla^2 V = \nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{3.110}$$

As we can see from Eq. (3.110), the Laplacian of a scalar function is a scalar. Expressions for $\nabla^2 V$ in cylindrical and spherical coordinates are given in Appendix C.

The Laplacian of a scalar can be used to define the Laplacian of a vector. For a vector $\mathbf{E}$ specified in Cartesian coordinates as

$$\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z, \tag{3.111}$$

the Laplacian of $\mathbf{E}$ is

$$\nabla^2\mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} = \hat{\mathbf{x}}\nabla^2 E_x + \hat{\mathbf{y}}\nabla^2 E_y + \hat{\mathbf{z}}\nabla^2 E_z. \tag{3.112}$$

Thus, in Cartesian coordinates the Laplacian of a vector is a vector whose components are equal to the Laplacians of the vector components. Through direct substitution, it can be shown that

$$\nabla^2\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}). \tag{3.113}$$

Chapter 1 Summary

Concepts

- Vector algebra governs the laws of addition, subtraction, and multiplication of vectors, and vector calculus encompasses the laws of differentiation and integration of vectors.
- In a right-handed orthogonal coordinate system, the three base vectors are mutually perpendicular to each other at any point in space, and the cyclic relations governing the cross products of the base vectors obey the right-hand rule.
- The dot product of two vectors produces a scalar, whereas the cross product of two vectors produces another vector.
- A vector expressed in a given coordinate system can be expressed in another coordinate system through the use of transformation relations linking the two coordinate systems.
- The fundamental differential functions in vector calculus are the gradient, the divergence, and the curl.
- The gradient of a scalar function is a vector whose magnitude is equal to the maximum rate of increasing change of the scalar function per unit distance, and its direction is along the direction of maximum increase.
- The divergence of a vector field is a measure of the net outward flux per unit volume through a closed surface surrounding the unit volume.
- The divergence theorem transforms the volume integral of the divergence of a vector field into a surface integral of the field's flux through a closed surface surrounding the volume.
- The curl of a vector field is a measure of the circulation of the vector field per unit area Δs, with the orientation of Δs chosen such that the circulation is maximum.
- Stokes's theorem transforms the surface integral of the curl of a vector field into a line integral of the field over a contour that bounds the surface.
- The Laplacian of a scalar function is defined as the divergence of the gradient of that function.

Mathematical and Physical Models

Distance Between Two Points

$$d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$d = \left[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$

$$d = \left\{R_2^2 + R_1^2 - 2R_1 R_2 [\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\right\}^{1/2}$$

Coordinate Systems — Table 3-1

Coordinate Transformations — Table 3-2

Vector Products

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Vector Operators

$$\nabla T = \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z}$$

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(see Appendix C for cylindrical and spherical coordinates)

Divergence Theorem

$$\int_{\upsilon} \nabla \cdot \mathbf{E} \, d\upsilon = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Stokes's Theorem

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

Important Terms

Provide definitions or explain the meaning of the following terms:

azimuth angle
base vectors
Cartesian coordinate system
circulation of a vector
conservative field
cross product
curl operator
cylindrical coordinate system
differential area vector
differential length vector
differential volume
directional derivative
distance vector
divergence operator
divergence theorem
divergenceless
dot product
field lines
flux density
flux lines
gradient operator
irrotational field
Laplacian operator
magnitude
orthogonal coordinate system
position vector
radial distance r
range R
right-hand rule
scalar product
scalar quantity
simple product
solenoidal field
spherical coordinate system
Stokes's theorem
vector product
vector quantity
unit vector
zenith angle

QUESTIONS

1 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, the position vector of point (x, y, z) and $r = |\mathbf{r}|$, which of the following is incorrect?

(a) $\nabla r = \mathbf{r}/r$

(b) $\nabla \cdot \mathbf{r} = 1$

(c) $\nabla^2(\mathbf{r} \cdot \mathbf{r}) = 6$

(d) $\nabla \times \mathbf{r} = \mathbf{0}$

2 Which of the following is mathematically defined?

(a) $\nabla \times \nabla \cdot \mathbf{A}$

(b) $\nabla \cdot (\nabla \cdot \mathbf{A})$

(c) $\nabla\, (\nabla V)$

(d) $\nabla\, (\nabla \cdot \mathbf{A})$

3 Which of the following is zero?

(a) grad div

(b) div grad

(c) curl grad

(d) curl curl

4 Stokes's theorem is applicable only when a closed path exists and the vector field and its derivatives are continuous within the path.

(a) True

(b) False

(c) Not necessarily

5 If a vector field $\mathbf{Q}$ is solenoidal, which of these is true?

(a) $\oint_L \mathbf{Q} \cdot d\mathbf{l} = 0$

(b) $\oint_S \mathbf{Q} \cdot d\mathbf{S} = 0$

(c) $\nabla \times \mathbf{Q} = \mathbf{0}$

(d) $\nabla \times \mathbf{Q} \neq \mathbf{0}$

(e) $\nabla^2\mathbf{Q} = \mathbf{0}$

6 Calculate the gradient of:

(a) $V_1 = 6xy - 2xz + z$ (b) $V_2 = 10\rho \cos \phi - \rho z$ (c) $V_3 = \dfrac{2}{r} \cos \phi$

7 Evaluate the divergence of the following vector fields:

(a) $\mathbf{A} = xy\mathbf{a}_x + y^2\mathbf{a}_y - xz\mathbf{a}_z$

(b) $\mathbf{B} = \rho z^2\mathbf{a}_\rho + \rho \sin^2 \phi\, \mathbf{a}_\phi + 2\rho z \sin^2 \phi\, \mathbf{a}_z$

(c) $\mathbf{C} = r\mathbf{a}_r + r \cos^2 \theta\, \mathbf{a}_\phi$

8 Evaluate $\nabla \times \mathbf{A}$ and $\nabla \cdot (\nabla \times \mathbf{A})$ if:

(a) $\mathbf{A} = x^2y\mathbf{a}_x + y^2z\mathbf{a}_y - 2xz\mathbf{a}_z$

(b) $\mathbf{A} = \rho^2z\mathbf{a}_\rho + \rho^3\mathbf{a}_\phi + 3\rho z^2\mathbf{a}_z$

(c) $\mathbf{A} = \dfrac{\sin \phi}{r^2}\mathbf{a}_r - \dfrac{\cos \phi}{r^2}\mathbf{a}_\theta$

9 Find $\nabla^2 V$ for each of the following scalar fields:

(a) $V_1 = x^3 + y^3 + z^3$

(b) $V_2 = \rho z^2 \sin 2\phi$

(c) $V_3 = r^2(1 + \cos \theta \sin \phi)$

10 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ is the position vector of point (x, y, z), $r = |\mathbf{r}|$, show that:

(a) $\nabla(\ln r) = \dfrac{\mathbf{r}}{r^2}$

(b) $\nabla^2(\ln r) = \dfrac{1}{r^2}$

Appendix C
Mathematical Formulas

Gradient, Divergence, Curl, and Laplacian Operators

Cartesian (Rectangular) Coordinates $(x,\ y,\ z)$

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\
&\quad + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)
\end{aligned}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates $(r,\ \phi,\ z)$

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} r & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) \\
&\quad + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) \\
&\quad + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]
\end{aligned}$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates $(R,\ \theta,\ \phi)$

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) \\
&\quad + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) \\
&\quad + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}
\end{aligned}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} R & \hat{\boldsymbol{\phi}} R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix} = \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$